2/2/23 MATH 2060B Tutorial

Announcements:

- HW2 Due 10 Feb 11:00 am.

Section 6.3: L'Hopital's Rules

Wants to handle

$\lim_{x \to a} \frac{f(x)}{g(x)}$ when they have indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, 0^0, $\infty - \infty$

Thm 6.3.1: f, g defined on $[a,b]$, $f(a) = 0 = g(a)$, $g(x) \neq 0$ for $x \in (a,b)$. If f, g are differentiable at a, with $g'(a) \neq 0$, then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Thm 6.3.3 L'Hopital's Rule I: f, g are differentiable on (a,b), $g'(x) \neq 0$ for all $x \in (a,b)$, with $\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x)$, then

if $\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$ exists or is $-\infty, +\infty$, then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)} \in \mathbb{R} \cup \{-\infty, +\infty\}$.

L'Hopital's Rule II: f, g differentiable on (a,b), $g'(x) \neq 0$ on (a,b), $\lim_{x \to a^+} g(x) = +\infty$, then

if $\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$ exists or is $-\infty, +\infty$, then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)} \in \mathbb{R} \cup \{-\infty, +\infty\}$.

Also works for left-handed and two-sided limits or if $a = -\infty$, $b = +\infty$.

Quick Examples:

1) $$\lim_{x \to \infty} \frac{\ln x}{x} \overset{LHR}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0.$$

"$\frac{\infty}{\infty}$"

2) $$\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right) = \lim_{x \to 0^+} \frac{\sin x - x}{x \sin x} \overset{LHR}{=} \lim_{x \to 0^+} \frac{\cos x - 1}{\sin x + x\cos x} \overset{LHR}{=} \lim_{x \to 0} \frac{-\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0.$$

"$\infty - \infty$"

3) $$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} \overset{LHR}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$$

"$0 \cdot -\infty$"

Nonexamples

Q3: $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \in (0,1] \\ 0 & x = 0 \end{cases}$, $g(x) = x^2$ on $[0,1]$.

Show that $\lim_{x \to 0} f(x) = 0 = \lim_{x \to 0} g(x)$ but that $\lim_{x \to 0} \frac{f(x)}{g(x)}$ DNE.

Pf: Clearly $\lim_{x \to 0} g(x) = \lim_{x \to 0} x^2 = 0$

$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$ by squeeze thm $(|f(x)| \leq x^2)$.

On $x \in (0,1]$, $\frac{f(x)}{g(x)} = \sin\left(\frac{1}{x}\right)$.

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ DNE: $x_n = \frac{1}{2n\pi} \to 0$ as $n \to \infty$, $\lim_{n \to \infty} \sin(2n\pi) = 0$

$y_n = \frac{1}{\left(\frac{\pi}{2} + 2n\pi\right)} \to 0$ as $n \to \infty$, $\lim_{n \to \infty} \sin\left(\frac{\pi}{2} + 2n\pi\right) = 1$.

Cannot apply Thm 6.3.1 because $g'(0) = 0$.

Cannot apply Thm 6.3.3 because $\lim_{x\to 0}\frac{f'(x)}{g'(x)}$ DNE:

$$\frac{f'(x)}{g'(x)} = \frac{2x\sin(\frac{1}{x})-\cos(\frac{1}{x})}{2x} = \sin(\tfrac{1}{x}) - \frac{\cos(\frac{1}{x})}{2x},$$ clearly $\lim_{x\to 0}\frac{f'(x)}{g'(x)}$ DNE.

<u>Q4</u>: $f(x)=\begin{cases} x^2 & x\in\mathbb{Q} \\ 0 & x\in\mathbb{R}\setminus\mathbb{Q}. \end{cases}$ $g(x)=\sin x$.

Use Thm 6.3.1 to show $\lim_{x\to 0}\frac{f(x)}{g(x)}=0$, explain why Thm 6.3.3 cannot be used.

<u>Pf</u>: $f(0)=0=g(0)$, $g'(0)=\cos 0=1\neq 0$, $g(x)\neq 0$ for $x\in(0,\varepsilon)$, ε small enough.

By Thm 6.3.1 $\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)} = f'(0)$

$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{f(x)}{x}$. Let $\varepsilon>0$ be given. Then since $\left|\frac{f(x)}{x}\right|\leq\frac{x^2}{x}=x$, as long as $|x|<\varepsilon$, we have $\left|\frac{f(x)}{x}\right|\leq|x|<\varepsilon$. So $\lim_{x\to 0}\frac{f(x)}{x}=0$.

c) $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = 0.$

Cannot use Thm 6.3.3, since $f'(x)$ DNE for $x \neq 0$.

Let $a \neq 0$, then $\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a} = \begin{cases} \lim\limits_{x\to a} \dfrac{f(x)-a^2}{x-a} & a \in \mathbb{Q} \\ \lim\limits_{x\to a} \dfrac{f(x)}{x-a} & a \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$

Pick x_n rationals $\to a$ as $n \to \infty$
y_n irrationals $\to a$ as $n \to \infty$ and check that they do not agree.

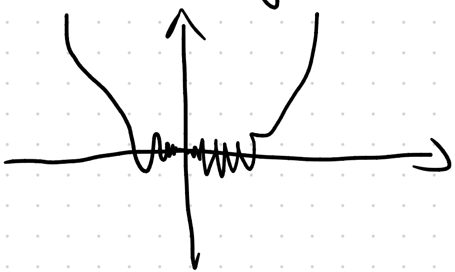
Q5: $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$, $g(x) = \sin x$.

Show that $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = 0$ but that $\lim\limits_{x\to 0} \dfrac{f'(x)}{g'(x)}$ DNE.

Pf: $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0} \dfrac{x^2 \sin(\frac{1}{x})}{\sin x} = \lim\limits_{x\to 0} \dfrac{x}{\sin x} x\sin(\frac{1}{x}) = \overset{1}{\lim\limits_{x\to 0} \dfrac{x}{\sin x}} \; \overset{0 \text{ squeeze thm}}{\lim\limits_{x\to 0} x\sin(\frac{1}{x})} = 0$

$$\frac{f'(x)}{g'(x)} = \frac{2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)}{\cos x} = \frac{2x\sin\left(\frac{1}{x}\right)}{\cos x} - \frac{\cos\left(\frac{1}{x}\right)}{\cos(x)}.$$

graph of $\frac{f(x)}{g(x)}$



Picking $x_n = \frac{1}{2n\pi} \to 0$ as $n \to \infty$, and

$$\lim_{n\to\infty} \frac{f'(x_n)}{g'(x_n)} = \lim_{n\to\infty} \frac{\frac{2}{2n\pi}\cancelto{0}{\sin(2n\pi)}}{\underset{\neq 0}{\cos\left(\frac{1}{2n\pi}\right)}} - \frac{\cancelto{1}{\cos(2n\pi)}}{\underset{\neq 0}{\cos\left(\frac{1}{2n\pi}\right)}} = -1.$$

$y_n := \frac{1}{\left(\frac{\pi}{2} + 2n\pi\right)} \to 0$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{f'(y_n)}{g'(y_n)} = \lim_{n\to\infty} \frac{\frac{2}{\left(\frac{\pi}{2}+2n\pi\right)}\cancelto{1}{\sin\left(\frac{\pi}{2}+2n\pi\right)}}{\underset{\neq 0}{\cos\left(\frac{1}{\frac{\pi}{2}+2n\pi}\right)}} - \frac{\cancelto{0}{\cos\left(\frac{\pi}{2}+2n\pi\right)}}{\underset{\neq 0}{\cos\left(\frac{1}{\frac{\pi}{2}+2n\pi}\right)}} = \lim_{n\to\infty} \frac{2}{\left(\frac{\pi}{2}+2n\pi\right)} \frac{1}{\cos\left(\frac{1}{\frac{\pi}{2}+2n\pi}\right)} = 0.$$